5.2 Measuring Momentum Through Traversal-Distance

The measurement of traversal-distance may be considered the space-time *dual* of the measurement of traversal time: instead of fixing x_1 and x_2 and measuring $t_F = t_2 - t_1$, one fixes t_1 and t_2 and measures $x_F = x_2 - x_1$. It is instructive to examine first this simpler case of traversal-distance and point out the similarities and the differences.

Unlike the case of traversal time, a measurement of traversal-distance can be described by the standard von Neumann interaction. For a free particle the Hamiltonian is

$$\mathbf{H} = \mathbf{p}^2$$

uncertainty relations, because $\mathbf{p}$ commutes ith $\mathbf{x}_F$, and $\mathbf{x}$ remains completely uncertain. Similarly, in the case of the traversal time e shall see that the measurement determines also the momentum during the traversal, ho ever unlike the present case, since the particle has to be in the interval $x_2 - x_1$ during the traversal, it is also a measurement of the location. This indicates that, in the latter case, in order not to violate the kinematic uncertainty principle for x and p, the accuracy ith hich the traversal time, T_F, or the momentum may be measured must be limited.

5.3 Measuring Traversal Time

For traversal time the classical equations of motion suggest that a traversal time operator might be given by

$$\mathbf{T_F} = \frac{mL}{\mathbf{p}}, \tag{5.149}$$

here $L = x_2 - x_1$. This operator is self-adjoint, but like the time-of-arrival operator, e shall see that different outcomes are found in a direct measurement of T_F and a measurement of the operator $\mathbf{T_F}$. Furthermore, one can measure the quantity $\mathbf{T_F}$ at any time, so there is no reason to believe that the particle actually traveled bet een the t o points in the time t_F. Since $\mathbf{T_F}$ is only a function of

here $\mathbf{p}'$ is the particle's momentum during the measurement, and $\mathbf{p}$ is the undisturbed momentum. Ho ever if the interaction is eak $Q \ll E_p$, then after a sufficient time, the clock ill read the undisturbed traversal time

$$\begin{aligned} \mathbf{P}(t \to \infty) - \mathbf{P}(0) &\simeq \int_0^\infty V\left(\mathbf{x}(0) - \frac{\mathbf{p}_o t}{m}\right) dt \\ &= \frac{m(x_2 - x_1)}{\mathbf{p}} \end{aligned} \tag{5.157}$$

If e require an accurate measurement of the traversal time, then a small dP ill result in large values of the coupling Q. If Q is too large, the clock can reflect the particle at x_1 and one ill obtain a traversal time equal to 0. This therefore imposes a restriction on the accuracy ith hich one can measure the traversal time. As in Chapter 3 e find that

$$\delta T_F > 1/E_p \tag{5.158}$$

is required in order to be able to measure the traversal time, and

$$\delta T_F \gg 1/E_p \tag{5.159}$$

in order to measure the undisturbed value of the traversal time.

Let us sho that the above conditions are consistent ith the uncertainty relations for the position and momentum. If (5.159) is satisfied, e have $Q \ll E$, and by eq. (5.156) the momentum during the measurement is

$$\mathbf{p}' \simeq \mathbf{p} - \frac{m}{\mathbf{p}}\mathbf{Q}. \tag{5.160}$$

Thus during the measurement, the momentum ill be uncertain by an amount

$$dp' \simeq \frac{m}{p_o} dQ. \tag{5.161}$$

In order to kno hether the particle entered our detector, e need to be able to distinguish bet een the case here the pointer is at its initial position $P = 0$, and the case here the particle has gone through the detector $P = t_F = {}^{mL}$

value of the pointer. The measured traversal time is then proportional to $P_f - P_i$. The relative accuracy of the traversal time ill then be given by $\delta T_f / T_f - dP/(P_f - P_i)$

Another condition e ill impose is that the inaccuracy of the measurement, δT_F, is less than the quantity e are trying to measure $\mathbf{T}_F$ (i.e. e are looking at accurate measurements). Finally, e assume that the experimentalist has no kno ledge of the state of the particle, and thus must set the initial state of the measuring device (and its inaccuracy δP) ith no prior kno ledge of the ensemble.

Before proceeding ith the argument, e should be clear to distinguish bet een different types of uncertainties. For an operator $\mathbf{A}$, there exists a kinematic uncertainty hich e ill denote by $d\mathbf{A}$ given by

$$d\mathbf{A}^2 = \langle \mathbf{A}^2 \rangle - \langle \mathbf{A} \rangle^2. \tag{5.164}$$

This is the uncertainty in the distribution of the observable A. There is also the inherent *inaccuracy* in the measuring device. This is the relevant quantity in equations (5.14 i [illegible]

[illegible]

The Heisenberg uncertainty relationship $dAdB > 1$ applies to measurements on ensembles. Given an ensemble, e measure **A** on half the ensemble and **B** on the other half. The uncertainty relation also applies to simultaneous measurements [2]. If e measure **A** and **B** simultaneously on each system in the ensemble, then the distributions of **A** and **B** must still satisfy the uncertainty relationship.

Returning no to the traversal time, e see that it can be interpreted as a simultaneous measurement of position and momentum. We kno the particle's momentum p during the time that it as bet een $x = x_1$ and $x = x_2$ from the classical equations of motion

$$t_F = \frac{mL}{p}. \tag{5.168}$$

In other ords, eigenstates of momentum must have traversal times given by equation (5.168). This measurement of momentum is analogous to the measurement described in Section 5.2. Instead of measuring the change in position at t o specified times t_1 and t_2, e are no measuring the difference in arrival times after specifying t o different positions x_1 and x_2. During the measurement, e also kno that the particle is some here bet een $x = x_1$, and $x = x_2$. i.e.. e kno that $x = \frac{x_1+x_2}{2} \pm L/2$.

The uncertainty relationship also applies to these measured quantities $\Delta x \Delta p > 1$. This essentially means that a detector of size L ill disturb the momentum by at least $2/L$, so that repeated measurements on an ensemble ill give $\Delta p > 2/L$. The position of the detector **X** commutes ith the momentum of the particle **p** [10] ho ever, e demand that the particle actually travel the distance L. The particle must actually be inside the detector during the measurement. As a result, **X** must be coupled to the position **x** of

[2] For a discussion of how the uncertainty relation applies to simultaneous measurements, see for example, Arthurs and Kelly[38] They propose a model for simultaneous measurements using a Hamiltonian $\mathbf{H} = \delta(t)(\mathbf{P}_1\mathbf{A} + \mathbf{P}_2\mathbf{B})$ which measures the variables **A** and **B** using two measurement pointers $\mathbf{Q}_1$ and $\mathbf{Q}_2$ which are conjugate to $\mathbf{P}_1$ and $\mathbf{P}_2$. They show how the Heisenberg uncertainty relation applies to the uncertainty in the outcomes of the measurement of **A** and **B**.

the particle and so a measurement of $\mathbf{X}$ is also a measurement of $\mathbf{x}$. This is hat e mean by a local interaction.

We no sho hy e expect (5.143) to be true. During the measurement of traversal and does not depend on the nature of the ensemble upon hich e ill be making measurements. For a free Hamiltonian, a measurement of the traversal time ill result in a final pointer position given by

$$P_f = P_o + \frac{mL}{p} \tag{5.172}$$

here p is the momentum of the particle in the absence of any measuring device. For eigenstates of $\mathbf{p}$ (or states peaked highly in p), e demand that the traversal time be given by the classically expected value [3]. Recall that the kinematic spread in the particle's momentum is given by $d\mathbf{p}^2 = \langle \mathbf{p}^2 \rangle - \langle \mathbf{p} \rangle^2$. A measurement of the traversal time for a particular experiment i can take on the values

$$\begin{aligned} T_i &= P_f \\ &= \frac{mL}{p} + \epsilon \end{aligned} \tag{5.173}$$

A given measurement T_i ill allo us to infer the momentum of the particle p_i during the measurement

$$p_i(T_i) = \frac{mL}{T_i} = \frac{mLp}{mL + p\epsilon}. \tag{5.174}$$

The average value of any po er α of the measured momentum is

$$\langle p_M \rangle = \int \left(\frac{mLp}{mL + p\epsilon} \right) \; f(p)g(\epsilon)dpd\epsilon \tag{5.175}$$

here $f(p)$ gives the distribution of the particle's momentum and $g(\epsilon)$ is the distribution of the fluctuations. We no choose the mass m of the ensemble so that e al ays have

$$\epsilon p \ll mL. \tag{5.176}$$

In other ords, e consider measurements on ensembles here the measurement is much more accurate than the quantity being measured. i.e.. $\delta T_F << T_F$. Indeed for the

[3] It is possible to include small deviations from the classical value, by including an additional term in (5.172). These fluctuations need to average to zero in order to satisfy the correspondence principle. For small fluctuations, the following discussion is not altered.

example given in the previous section, for every given then implies

$$\delta T_F^2 > \frac{1 - \frac{1}{4}L^2 dp^2}{\langle E\rangle^2 + dE^2}. \tag{5.186}$$

No e note that e can arrange our experiment ith Ldp arbitrarily small, by choosing dp of the ensemble arbitrarily small. i.e.. the uncertainty in the traversal time is small. As a result, in order to ensure that Heisenberg's uncertainty relation is never violated, e must have

$$\delta T_F > \frac{1}{\sqrt{\langle E\rangle^2 + dE^2}}. \tag{5.187}$$

Since dp is small, e can rite

$$\delta T_F > \frac{1}{\langle E\rangle}. \tag{5.188}$$

It is interesting to note that since the momentum operator commutes ith the free Hamiltonian, the restriction on traversal time measurements only comes from the dynamical considerations given above.

5.5 From Traversal Time to Barrier Tunneling Time

We have seen that the measurement of the traversal time given t o positions cannot be made arbitrarily accurate. We have argued this by looking at a simple model for measuring traversal time, and e have also given a model independent, qualitative proof of this hich applies hen the measurement only disturbs the system slightly. Finally e have given a more rigorous argument hich applies hen the uncertainty in traversal time is small. This strongly suggests that the limitation on measurements of arrival times is a general rule and not just an artifact of the types of models considered so far. Operators for both the traversal time and the arrival time don't seem to correspond to physical (continuous) processes. The case of traversal time is different from time-of-arrival in that there does exist a self-adjoint traversal time operator, and the semi-bounded spectrum of the Hamiltonian does not seem to play an important role in the restriction on measurement accuracy. The accuracy restriction on traversal time may be particularly important for experiments on barrier tunneling time. If one uses a physical clock to measure the time it takes for a particle to travel from one location to another, ith a barrier situated some here bet een the t o locations [17][32], then the accuracy of this clock may affect the tunneling particle. The limitation presented in this Chapter seem to indicate that measurements of barrier tunneling times ould also need to be inherently inaccurate, because if one tries to measure the tunneling time too accurately, the particle may be unable to tunnel. Our result concerning traversal time indicates that the barrier tunneling time also cannot be precisely defined.