for some C C k M 0. Also, $\in C^{\infty}$ has the properties

$$| \ {}^{2}_{ij} \ | \leq C_{ij}\langle x\rangle^{k} \qquad |\nabla \ |^{2} \geq 4(\ + \ {}_{0}) \tag{1.4}$$

for some C_{ij} ${}_{0}$ N 0. Finally, 0.

Remark: All of our results hold for weaker assumptions on the growth of V and , however (1.3) and (1.4) are convenient for our purposes.

We will show in section 2 that for $V(x)$ and as in (1.3) and (1.4) respectively, the linearized problem is spectrally stable, that is, the spectrum is bounded away from $\operatorname{Re} z \geq 0$ uniformly in h. Yet, we also show that (1.1) has an unstable equilibrium at $u \equiv 0$ for all potentials $V(x)$ satisfying (1.3) and all satisfying (1.4). Specifically, we show

Theorem 1.

are poor, we are unable to exhibit blow-up starting from a quasimode. Instead, we present a simple and explicit construction of quasimodes for $P(x\ hD)$ (for a more general setting see [5]). We then use these quasimodes as initial data in numerical simulations and observe that, although in some cases the ansatz solution blows up more quickly, the solutions with quasimode initial data behave similarly to what is expected from a pure eigenvalue for (1.2) with positive real part.

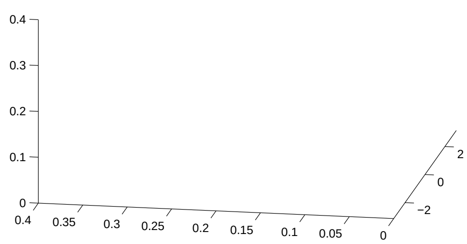

Figure 1. The plot shows numerical simulations of the evolution of (1.1) with $h = 1\ 193$ and two initial data. The evolution with initial data a real valued $O(h^3)$ error quasimode with eigenvalue $z = \frac{1}{16}$ is shown in the top two graphs and that with the ansatz constructed in the proof of Lemma 3 as initial data is shown in the bottom two graphs. We observe that, when the initial data is a quasimode, blowup occurs in time

of quasimodes for one dimensional problems. Although the results are known, (see [3],[5],[12]) a self-contained presentation is useful since we need the quasimodes for our numerical experiments. Also, there is no reference in which analytic potentials (for which quasimodes have $O(\exp(-c\ h))$ accuracy) is treated by elementary methods in one dimension. Section 4 is devoted to the proof of Theorem 1 using heat equation methods. Finally, in Section 5 we report on some numerical experiments which suggest that quasimode initial data gives more natural blow-up and that blow-up occurs at complex energies.

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2. Spectrum and Pseudospectrum

We do not use the results of this section to prove Theorem 1. Instead, we present them to emphasize the connection of the size of the resolvent with instability. We believe that instability based on quasimodes would be more natural and allow for proof of instability at complex energies. We illustrate this with numerics in Section 5.

To describe the spectrum of $P(x\ hD)$, we observe that

$$e^{\frac{(x)}{}}$$

then $z \in \ (Q)$. We use this condition to show that the pseudospectrum of $P(x\ hD)$ nontrivially interesects the right half plane. Specifically,

Lemma 1. *For $P(x\ hD)$ given by (1.2),* $\overline{\ (P(x\ hD))} \bigcap \{\operatorname{Im} z = 0\} = (-\infty\ \ ]$.

Proof. First, observe that

$$P(x\ \ ) = -|\ \ |^2 + i\langle\nabla\ \ \rangle - V(x) + \quad \text{and} \quad \{\operatorname{Re} P\ \operatorname{Im} P\} = -2\langle\ ^2\ \ \rangle + \langle\nabla V\ \nabla\ \ \rangle$$

We have assumed $\operatorname{Im} z = 0$. Therefore, we need only show that, for a dense subset $U \subset (-\infty\ \ ]$, $y \in U$ implies that there exists x such that (2.1) holds for the symbol $P(x\ \ )$, at $(x\ 0)$ with $z = y$.

We proceed by contradiction. Suppose there is no such U. Then, there exists $O \subset [0\ \infty)$ open such that for all $x \in V^{-1}(O)$, $\langle\nabla V\ \nabla\ \ \rangle(x\ 0) \geq 0$. Let $\ _t := \exp(ti\langle\nabla\ \ D\rangle)$ be the integral flow of $i\langle\nabla\ \ D\rangle$ and $x_0 \in \mathbb{R}^d$ have $V(x_0) = 0$. Define $f(t) := V(\ _t(x_0))$. Then $\ _t f = \langle\nabla V(\ _t(x_0))\ \nabla\ (\ _t(x_0))\rangle$.

Suppose that $\ _t(x_0)$ escapes every compact set as $|t|$ increases. Then (1.4) implies that $f(t) \to \infty$ as $|t|$ increases. Let $w \in O$ and $t_0 := \inf\ t \in \mathbb{R} : f(t) = w$. Then t_0 is finite since $w \geq f(0)$ and $f(t) \to \infty$. Together, $f(t_0) = w \in O$ and $f^{-1}(O)$ open imply the existance of $\quad 0$ such that for $t \in (t_0 -\ \ t_0 +\ \ )$, $f(t) \in O$. But, $f(t) \in O$ implies $f'(t) \geq 0$. Therefore, $f(t) \leq w$ for $t \in (t_0 -\ \ t_0)$ and thus, since $f(t) \to \infty$, there exists $t\ \ t_0$ such that $f(t) = w$, a contradiction.

We have shown that there is a dense subset $U \subset (-\infty\ \ ]$ with $U \subset\ (P)$. Hence $(-\infty\ \ ] \subset \overline{\ (P)}$. Next, observe that $\sup \operatorname{Re} P(x\ \ ) = \ $ and thus, $\overline{\ (P(x\ hD))} \bigcap \{\operatorname{Im} z = 0\} = (-\infty$

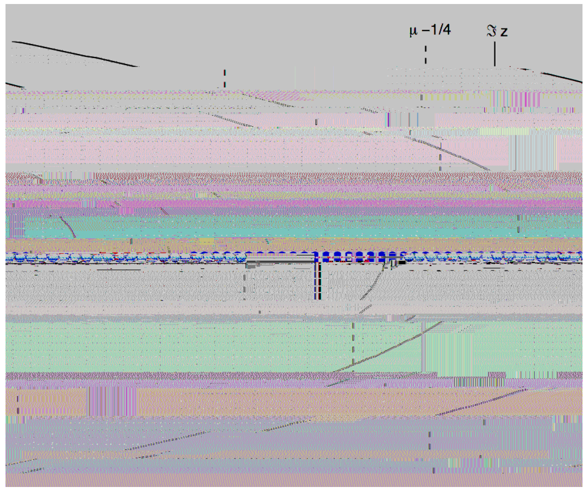

Figure 2. We see that the spectrum of (1.5) (blue dots) is bounded away from Re z = 0, while the pseudospectrum (shaded region) enters the right half plane. The region for which we prove blow-up corresponds to the dashed red line.

3. One Dimensional Quasimodes

We proceed by constructing quasimodes for operators in the one dimensional case with $i\langle\nabla \quad D\rangle = \quad _x$. We implement WKB expansion for the quasimode following the method used in [3]. Let

$$P(x \ hD) := -(hD_x)^2 + ihD_x + V \tag{3.1}$$

where $V \in C^\infty$ and V may be complex.

Remark. ;

When V is real analytic than we can find such that

$$\|(P(x\ hD) - z)\ \|_{L^2} \leq C\exp(-1\ Ch)$$

Proof. Let $\in C_c^\infty(\mathbb{R})$ with $(x) = 1$ if $|x|$ 2 and $(x) = 0$ if $|x|$ where will be determined below. Define $f := \exp(i\ \ h)a(x)$ where

$$a(x) = \sum_{m=0}^{N-2} a_m(x)h^m$$

Finally, let $g(x_0 + x) := (x)f(x)$ for all $x \in \mathbb{R}$.

We will find appropriate a_m and in what follows. First, by a simple computation

$$(P(x\ hD) - z)\ f = \left(\sum_{m=0}^{N} h^m \ \ _m\right) e^{i\ \ h}$$

where $_m$ are inductively defined by

$$_m := (-(\ ')^2 + i\ ' + V - z)a_m + i\ ''a_{m-1} + (2i\ ' + 1)a'_{m-1} + a''_{m-2}$$

where we use the convention that $a_m \equiv 0$ for m $N - 2$ or m 0. Now, we set $_m = 0$ for $0 \leq m \leq N - 1$. Given that is small enough, this will enable us to determine all a_m as well as .

Observe that, using the condition, $_0 = 0$, we obtain

$$'^2 - i\ ' = V - z$$

Now, letting $z = -\ _0^2 + i\ _0 + V(x_0)$, we have a complex eikonal equation

$$' - \frac{i}{2}\ ^2 = V(x_0 + x) - V(x_0) + \ \ _0 - \frac{i}{2}\ ^2$$

Then, letting $\tilde{\ } = \ - \frac{i}{2}x$, we have

$$\tilde{\ }(x) := \int_0^x \ V(x_0 + t) - V(x_0) + \ \ _0 - \frac{i}{2}\ ^2 !$$

for all small enough x and h. Also, for x and h small enough

$$:= (2i\ '+1)^{-1}$$

satisifies $|\ (x)| \leq$. We choose 0 small enough so that these conditions both hold for 0 h ^{2} and $|x|$.

The condition $_{m+1} = 0$, implies

$$a_m' = - (i\ '' a_m + a_{m-1}'')$$

with the convention that $a_{-1} \equiv 0$ and initial conditions,

$$a_0(0) = 1 \quad a_m(0) = 0 \quad m \quad 0$$

Putting $G(x) := \int_0^x i\ ''(y)\ (y)dy$ we obtain $a_0 = \exp(-G(x))$ and

$$a_{m+1}(x) := -e^{-G(x)} \int_0^x e^{G(y)}\ (y) a_m''(y) dy \quad m \quad 0 \tag{3.2}$$

Before proceeding to show exponential error for V analytic, we show $O(h^N)$ error for arbitrary V. To complete the proof of $O(h^N)$ quasimodes, we need to estimate

$$\|(P(x\ hD) - z)g\|_{L^2}\ \ \|g\|_{L^2}$$

Let C denote various positive constants that are independent of h and x. Then,

$$\begin{aligned} \|g\|_{L^2}^2 &\geq \int_{-\ 2}^{\ 2} |f(x)|^2 dx \geq \int_{-\ 2}^{\ 2} e^{-6\ x^2 h^{-1} - C} dx \\ &= \int_{-\ h^{-1\ 2}\ 2}^{\ h^{-1\ 2}\ 2} e^{-6\ t^2 - C} h^{1\ 2} dt \geq \int_{-1\ 2}^{\ 1\ 2} e^{-6\ t^2 - C} h^{1\ 2} dt = Ch^{1\ 2} \end{aligned} \tag{3.3}$$

Next, we compute

$$\begin{aligned} \|(P(x\ hD) - z)\, g\|_{L^2} &= \|h^2 f\ '' + 2h^2 f'\ ' + hf\ ' + \ (P(x\ hD) f - zf)\|_{L^2} \\ &\leq h^2 \|f\ ''\|_{L^2} + 2h^2 \|f'\ '\|_{L^2} + h\|f\ '\|_{L^2} + \|h^N\ \ _N e^{i\ \ h}\|_{L^2} \end{aligned} \tag{3.4}$$

Thus, we need to estimate each of the norms. Note that $'$ and $''$ have support in $\{x :\ \ 2 \leq |x| \leq\ \}$. Thus, we have

$$\|f\ ''\|_{L_2}^2 \leq C_3 \int_{2 \leq |x| \leq} e^{-2\ x^2 h^{-1} + C} dx \leq Ce^{-\ \ 2\ 2h} \tag{3.5}$$

Similarly,

$$\|f'\ '\|_{L^2}^2 \leq Ce^{-\ \ 2\ 2h} \quad \|f\ '\|_{L^2}^2 \leq Ce^{-\ \ 2\ 2h} \tag{3.6}$$

Next, observe that

$$\|h^N\ \ _N e^{i\ \ h}\|_{L^2}^2 \leq h^{2N} \|\ _N\|_{L^\infty}^2 \int$$

Now, $|\ _m| \le c_m$ on $\{x : |x| \le \ \}$, uniformly for $h \le \ ^2$

Next, we prove similar estimates for $^{p}_{w}(\ _0 b_m)$. By Leibniz rule, we have that

$$|\ ^{p}(\ _0 b_m)|\ _w = \sum_{k=0} \frac{p!}{k!(p-k)!}\ ^{k}\ _0\ ^{p-k} b_m\ _w \leq \sum_{k=0} C_0^{p+2} C_1^{m+1} r_{k\,m\,p} \tag{3.12}$$

where

$$r_{k\,m\,p} := \frac{p!}{k!(p-k)!} k^k (m+p-k)^{m+p-k}$$

We claim that for $0 \leq k \leq \frac{p}{2}$, $r_{k\,m\,p} \geq r_{p-k\,m\,p}$. To see this, we write this inequality as

$$k^k (m+p-k)^{m+p-k} \geq (p-k)^{p-k} (m+k)^{m+k} \qquad \text{for } 0 \leq k \leq \frac{p}{2}$$

Putting $x := \frac{k}{m}$ and $y = \frac{p-k}{m}$, the inequality is equivalent to

$$x^x (1+y)^{1+y} \geq y^y (1+x)^{1+x} \qquad \text{for } 0 \leq x \leq y$$

which follows from the monotonicity of the function $x \mapsto \frac{1+x}{x}^{\ x} (1+x)$

Next, observe that, $0 \leq k \quad p-1$,

$$\begin{aligned} \frac{r_{k+1\,m\,p}}{r_{k\,m\,p}} &= \frac{p-k}{k+1} \frac{(k+1)^{k+1}}{k^k} \frac{(m+p-k-1)^{m+p-k-1}}{(m+p-k)^{m+p-k}} \\ &= \frac{p-k}{m+p-k-1} \ 1 + \frac{1}{k}^{\ k}\ 1 - \frac{1}{m+p-k}^{\ !\, m+p-k} \\ &\leq \frac{p-k}{m+p-k-1} e^{1-1+\frac{1}{2(m+p-k)}} \leq e^{\frac{1}{2(m+p-k)}} \end{aligned}$$

where we use $\log(1-x) \leq -x + \frac{x^2}{2}$. Then, since for $0 \leq k \leq \frac{p}{2}$, $r_{k\,m\,p} \geq r_{p-k\,m\,p}$, we have

$$\begin{aligned} |\ ^{p}(\ _0 b_m)|\ _w &\leq 2C_0^{p+2} C_1^{m+1} \sum_{k=0}^{\frac{p}{2}+1} r_{k\,m\,p} \leq 2C_0^{p+2} C_1^{m+1} \sum_{k=0}^{\frac{p}{2}+1} r_{0\,m\,p} \prod_{n=0}^{k-1} e^{\frac{1}{2(m+p-n)}} \\ &\leq 2C_0^{p+2} C_1^{m+1} \sum_{k=0}^{\frac{p}{2}+1} r_{0\,m\,p} e^{\frac{k}{2m+p}} \leq 2C_0^{p+2} C_1^{m+1} \sum_{k=0}^{\frac{p}{2}+1} r_{0\,m\,p} e^{\frac{p+2}{4m+2p}} \\ &\leq C_0^{p+2} C_1^{m+1} (p+2) r_{0\,m\,p} e^{\frac{1}{2}} \leq e^{\frac{1}{2}} C_0^{p+2} C_1^{m+1} (m+p+1)^{m+p+1} \end{aligned}$$

Therefore, there exists $M_1 \quad 0$ such that

$$|\ ^{p}_{w}(\ _0 b_m)|\ _w \leq M_1 C_0^{p+2} C_1^{m+1} (m+p+1)^{m+p+1} \tag{3.13}$$

By analagous argument, there exists $M_2 \quad 0$ such that

(3.14) $\ ^{p}_{w}$ Z @ $z(p$ ^{p}Y1 $n=$

n

n $\ _{p,\,p} \leq 2C^{p+}$

Next, choose C_1 C_0 (4 max(M_1 M_2 1)) Then, combining (3.10), (3.11), (3.13), and (3.14), we have

$$| \ {}^{p}_{w} b_{m+1}(w)| \ {}_{w} \leq C_0^{p+1} C_1^{m+2} (m + p + 1)^{m+p+1}$$

Then, since $w \to z$ is a change of variables independent of m which maps ${}_{w} \to$ and $b_m(w) = d_m(z(w))$, we have

$$|d_m| \quad = |$$

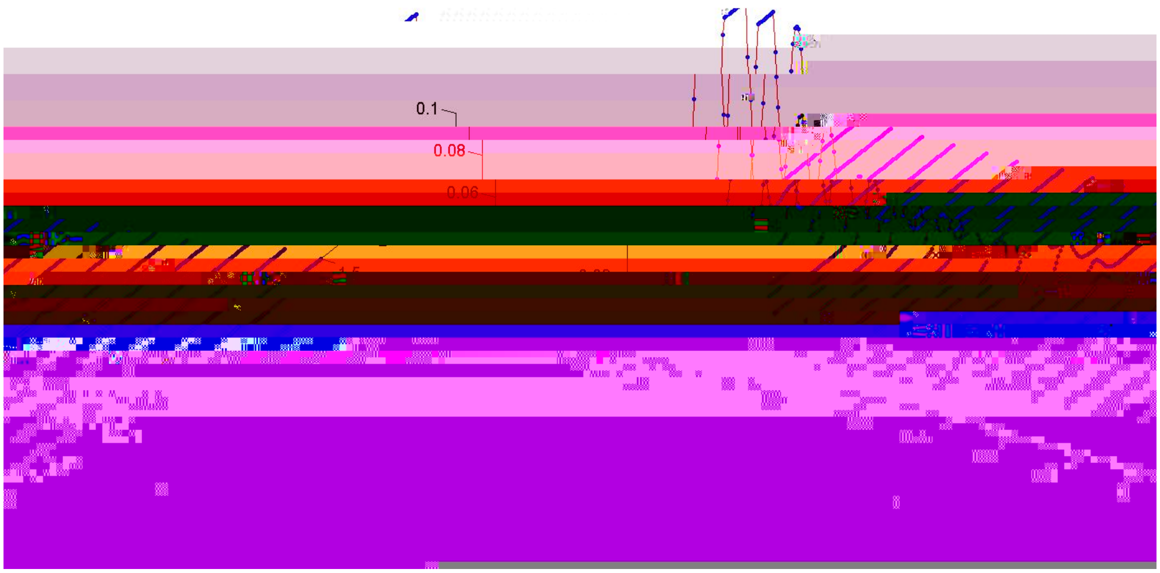

Figure 3. We set $h = 10^{-2}$ and see that the di erence between the solution to (1.1) with initial data a quasimode with error $O(h^2)$ (red line) and the solution with initial data a quasimode with error $O(h^3)$ (blue dots) is negligible. Thus, by using $O(h^3)$ error quasimodes, we have not introduced large error into our numerical calculations.

Lemma 3. *Fix* 0, , 0 $\leq \frac{1}{2}(\quad - \quad)$, *and* $(x_0 \ a \quad) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+$ *such that both* $_t(B(x_0 \ 2a)) \subset V^{-1}[0 \quad - \quad - \quad]$ *for* $t \leq$ *and* $_t$ *is defined on* $B(x_0 \ 2a)$ *for* $0 \leq t \quad 2$ *. Then, for each*

$$0 \quad h \quad h_0$$

where h_0 *is small enough, there exists*

$$u_0(x) \geq 0 \qquad \|u_0\|_{C^p} \leq \exp\left(-\frac{1}{Ch}\right) \qquad p = 0 \ 1$$

and 0 t_1 *so that the solution to (1.1) with initial data* u_0 *satisfies* $u(x \ t_1) \geq 1$ *on* $x \in \ _{t_1}(B(x_0 \ a))$.

Proof. Let solve

(4.1) $$(h \ _t - P(x \ hD)) \quad = 0 \quad (x \ 0) = \quad _0$$

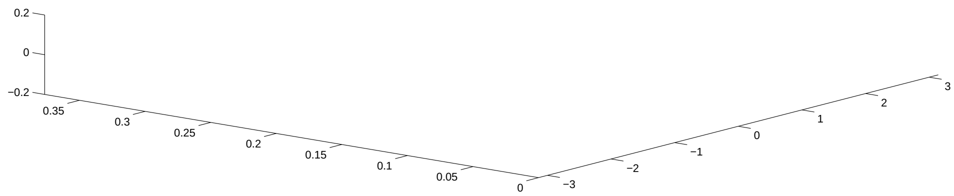

Figure 4. We show a numerical simulation $u(t)$ of the evolution of (1.1) with a quasimode at imaginary energy as initial data. Specifically, we set $h = 1\ 193$ and use a quasimode with eigenvalue $z = \frac{1}{16} + \frac{i}{4\sqrt{2}}$. The real part is shown in

Remark 1. If a shorter time is desired, one may use initial data of $O(h^n)$ to obtain the same result in time $O(h|\log h|)$.

Remark 2. Notice that to obtain a growing subsolution it was critical that 0. This corresponds precisely with the movement of the pseudospectrum of $P(x\ hD)$ into the right half plane.

Now, we will demonstrate finite time blow-up using the fact that in time $O(1)$ the solution to (1.1) is ≥ 1 on an open region. The proof of theorem 1 follows

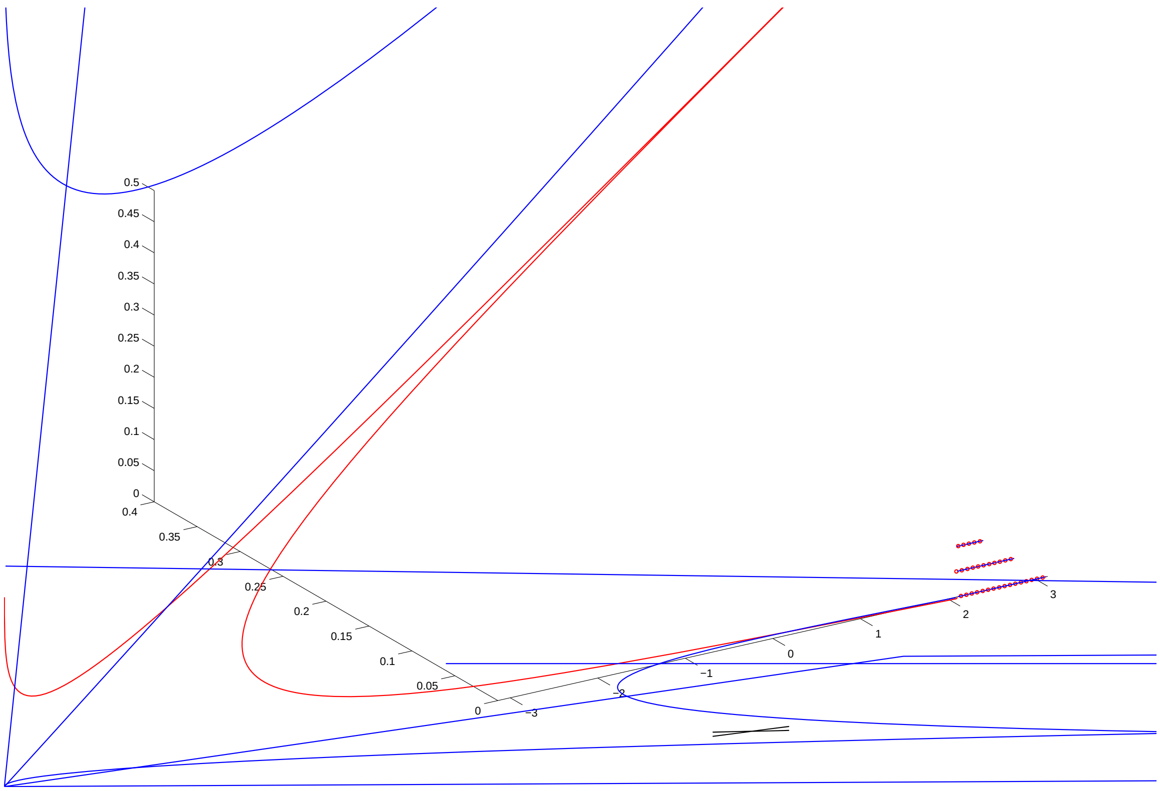

Figure 5. We show simulations of solutions to the equation $hu_t = P_1(x\ hD)u$ with $h = 1\ 193$. The solution using a quasimode u_0 with eigenvalue $z = \frac{1}{16}$ and error $O(h^3)$ as initial data is shown in the blue solid linse. The red dotted lines show $u(x\ t) = u_0(x)\exp(zt\ h)$. We see that the solution to the linearized problem (4.1) with quasimode initial data closesly approximates the exponential until $t \approx 0\ 3$

Proof. Let $u_0(x)$ and t_1 be the initial data and time found in Lemma 3 with $(a\ x_0\ \ )$ such that $_t$ is defined on $B(x_0\ a\ 0h$ $a=$

Next, let $y' = \quad _t$

Now, on t , we have $V(y') \leq \frac{}{2}$. Thus, for 0 t ,

$$h[v]_t \geq \frac{}{2}[v] + (1 - O(h^2))[v]^3 - O(h^2)$$

We have that $[v](t_1) \geq 1 \; 4$ and 0. Therefore there exists 0 such that, for h small enough and $t_1 \leq t \leq t_1 +$,

$$h[v]_t \geq \frac{}{4}[v] + \frac{1}{2}[v]^3$$

But, the solution to this equation with initial data $[v](0) \geq 1 \; 4$ blows up in time $t_2 = O(h)$. Hence, so long as $t_1 + t_2$ min($t_1 +$) and h is small enough, $[v]$ blows up in time $t_1 + t_2$. Observe that since t_1 , $0 \leq t_1 + t_2 = t_1 + O(h)$ min($t_1 +$) for h small enough. Thus, the solution to 1.1 blows up in time $O(1)$.

Remark. A similar result holds for polynomially small data with blow up in time $O(h|\log h|)$.to this eq2978 0 0

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