A Survey of Results in Poincaré Recurrence Estimation

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Abstract

The Poincaré Recurrence theorem states that any probability measure preserving map has almost everywhere recurrence. However, it gives no information on how quickly this recurrence occurs. In the last twenty years, significant advances have been made in tools for estimating the Poincaré recurrence times in measure preserving dynamical systems. These methods connect the long term behavior of recurrence times to the Hausdorff

2 Notation

We will use a combination of the notations used by Barreira and Boshernitzan that are summarized below.

Definition 1 *m_α is defined to be the Hausdorff-α measure on a metric space (X, d).*

Definition 2 *A* measure preserving system *(m.p.s.) is a probability space (X, N, μ) together with a measure preserving map $T : X \to X$.*

Definition 3 *A* metric measure preserving system *(m.m.p.s.) is an m.p.s. with a metric d such tath the open sets relative to d are in N.*

Definition 4 *The* self return time *of a point x to the ball $B(x, r)$ is*

$$\tau_r(x) = \inf\{n \in \mathbb{N} | d(T^n x, x) < r\}.$$

Definition 5 *The* return time *of a point $y \in B(x, r)$ to the ball $B(x, r)$ is*

$$\tau_r(y, x) = \inf\{n \in \mathbb{N} | d(T^n y, x) < r\}.$$

Definition 6 *The* lower *and* upper recurrence rates *of x are respectively*

$$R^l(x) = \liminf_{r\to 0} \frac{\log \tau_r}{-\log r} \quad R^u(x) = \limsup_{r\to 0} \frac{\log \tau_r}{-\log r}$$

Definition 7 *The* lower *and* upper pointwise dimensions *of μ at a point $x \in X$ are respectively*

$$d^l_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad d^u_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

Definition 8 *The* Hausdorff dimension *of a probability measure μ on X is*

$$\dim_H \mu = \inf\{\dim_H Z | \mu(Z) = 1\},$$

where $\dim_H Z$ is the Hausdorff dimension of $Z \subset X$.

Definition 9 *We say that the measure μ has* long return time *with respect to T if, for μ-almost every $x \in X$ and for $\epsilon > 0$ small enough,*

$$\liminf_{r\to 0} \frac{\log \mu(A_\epsilon(x,r))}{\log \mu(B(x,r))} > 1$$

where

$$A_\epsilon = \{x \in B(x, r) | \tau_r(y, x) \le \mu(B(x, r))^{-1+\epsilon}\}.$$

Definition 10 *We say that the measure μ is* weakly diametrically regular (w.d.r) *on a set $Z \subset X$ if $\exists \eta > 1$ such that for μ-almost every $x \in Z$ and every $\epsilon > 0$, $\exists \delta > 0$ such that if $r < \delta$ then*

$$\mu(B(x, \eta r)) \le \mu(B(x, r)) r^{-\epsilon}. \qquad (1)$$

We will now prove the following small consequence of 10.

Lemma 1 *If μ is w.d.r. on Z X, then $\eta > 1$ $\delta > 0$ for μ almost every x Z and every > 0 such that $r < \delta$ equation (1) holds.*

Proof:
Since μ is w.d.r, $\eta > 1$ such that for μ-almost everywhere x Z and every > 0, $\delta > 0$ such that $r < \delta$ (1).

Claim:

$$\mu(B(x, \eta^k r)) \leq \mu(B(x, r))r^{-k} \quad \text{for } r < \frac{\delta}{\eta^{k-1}}.$$

We proceed by induction. The base case $k = 1$ is clear. Assume that the inductive hypothesis is true for k − 3B0 Td $B(x, \eta$ [(.444(b)1n,e)-333w3(h)28(yp)-2ae the We(k k 1 0.9626 Tf 4.902 -3.614 (9) [(k) 5T1/F27 0.9626 Tf 4.771 0 Td

and, if $m_\alpha(Y) = 0$, *then*

$$\liminf_{n\to\infty}\{n^{\frac{1}{\alpha}} d(f(x), f(T^n x))\} = 0. \tag{3}$$

In other words, $\min_{1\leq n\leq N}\{d(f(x), f(T^n x))\} \approx (\frac{c}{N})^{\frac{1}{\alpha}}$ for some constant $c \geq 0$ and N large enough. We can easily adjust this theorem to answer a question about how quickly the map T recurs by taking f to be the identity. By doing so, we arrive at the following theorem of Boshernitzan.

Theorem 3 *Let* (X, N, μ, T, d) *be an m.m.p.s, If, for some* $\alpha > 0$, m_α *is* σ-*finite on* Y. *Then for* μ-*almost every* $x \quad X$

$$\liminf_{n\to\infty}\{n^{\frac{1}{\alpha}} d(x, T^n x)\} < \infty \tag{4}$$

and, if $m_\alpha(Y) = 0$, *then*

$$\liminf_{n\to\infty}\{n^{\frac{1}{\alpha}} d(x, T^n x)\} = 0. \tag{5}$$

We can see from these theorems that, to within a constant multiple, we can obtain an upper bound on the global long term behavior of the recurrence of any any at the Hausdorff dimension of the space Y.

These statements are remarkable since, apriori, one would expect that the specific nature of a map would play a large role in its recurrence times and, even if the nature of the map did not affect this upper bound significantly, one would expect that, since the space of measurable functions has widely varying behavior, different measurable functions f would have widely varying recurrence behavior. However, we can now see that no matter how different their behavior, two μ-preserving maps T, T' with measurable functions f, f' respectively must have the same upper bound on their behavior. Because of this, we might suspect that these upper bounds are very far from optimal, however, [4] is able to find examples where these bounds are in fact optimal. We will see in the work of Barreira in [1] that in some more restricted domains and families of maps we are able to obtain both upper and lower bounds on the recurrence behavior of the map T given only information about the measure structure on X.

4 Local Recurrence

In this section, we restrict our attention to Borel measurable transformations T on separable metric spaces X and, in many cases, $X \subset \mathbb{R}^d$. This appears to be a large restriction, however, by the Whitney embedding theorem if $X \subset M$ with M a finite dimensional smooth manifold then X can be smoothly embedded into $\mathbb{R}^d$ for some $d > 0$ and there we can apply the theorems in this section.

We will now discuss theorems that relate the upper and lower recurrence times to the pointwise dimension of the manifold X. Note that in this section, we will not discuss recurrence of measurable functions of an m.p.s., but will assume that X is an m.m.p.s. and discuss the recurrence of the map T.

In [1] Barreira is able to obtain local upper bounds on the lower and upper recurrence rates without additional assumptions on T or the space X. He does this in terms of the lower and upper pointwise dimensions of the space X.

Theorem 4 *Let (X, N, μ, T, d) be an m.m.p.s. with T Borel measurable and μ w.r.d., then for μ- almost every $x \in X$*

$$R^l \leq d^l_\mu \text{ and } R^u \leq d^u_\mu. \tag{6}$$

The theorem can be better understood in the following form. For r small enough we obtain $\tau_r(x, x) \geq r^{-d^l_\mu}$. This is obtained by applying the bound on R^l(xTf 5.229 -2.041 Td [(4.

By examinging $x \in X$ locally and looking only at the map T instead of measurable functions $f : X \to Y$, Barreira[1] improves the upper bound obtained from global considerations in the following theorem.

Theorem 5 *If (X, N, μ, T, d) is an m.m.p.s. with T a Borel measurable transformation and μ w.d.r. then (5) holds with f the identity $\forall \alpha > d^l_\mu(x)$.*

This statement is in fact stronger than that in theorem 3 since, using Young's criteria from [9], one can show $\dim_H X \geq d^l_\mu(x)$ for μ almost every $x \in X$. Thus, if we have information about μ locally, we will in general gain a better upper bound on the recurrence time for T by examining the local dimension of μ.

The final result that we will discuss tightens both the upper and lower bound using local information about the T invariant measure μ given additional information about the relationship between the measure μ and the map T.

Theorem 6 *[1] Let (X, N, μ, T, d) be an m.m.p.s. with T Borel measurable and μ w.d.r. If μ has long return time with respect to T, and $d^l_\mu(x) > 0$ for μ almost every $x \in X$, then for μ almost every $x \in X$*

$$R^l(x) = d^l_\mu(x) \text{ and } R^u(x) = d^u_\mu(x). \tag{7}$$

Thus, if μ has a long return time with respect to T, we can improve from the bounds

$$r^{-d^l_\mu} \leq \tau_r(x, x) \leq C_x r^{-\dim_H X}$$

to the bounds

$$r^{-d^l_\mu} \leq \tau_r(x, x) \leq r^{-d^u_\mu X}.$$

From a further theorem in [1] we see that the class of systems with long return time includes all those equilibrium measures supported on locally maximal hyperbolic sets.

Since these results seem to be very strong, one may be led to believe that measures that are w.d.r. are not very common, however, the following lemma shows that each one of these theorems applies to Borel measurable subsets of $\mathbb{R}^d$ for any finite d.

Lemma 3 *[1] Any Borel probability measure on $\mathbb{R}^d$ is w.d.r.*

Proof:
Let μ be a Borel probability measure on $\mathbb{R}^d$.
Claim: It is sufficient to show that for μ almost every $x \in \mathbb{R}^d$

$$\mu(B(x, 2^{-n})) \leq n^2 \mu(B(x, 2^{-n-1})) \tag{8}$$

for sufficiently large $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Let $2^{-n-2} r \leq 2^{-n-1}$. Then we have

$$\mu(B(x,2r)) \leq \mu(B(x,2^{-n})) \leq n^2\mu(b(x,2^{-n-1}))$$

for $n > N$ large enough i.e. $r < 2^{-N-1}$. But X R^d, we have

$$\frac{\mu(B(x,2r))}{\mu(B(x,2^{-n-1}))} \geq \frac{\mu(x,2^{-n-2})}{\mu(B(x,2^{-n-1}))} \geq (n-1)^{-2} \geq n^{-2}.$$

Thus, we obtain $\mu(B(x,2r)) \leq n^4\mu(B(x,r))$. Thus, we need only find $n > 0$ such that $n^4 \leq r^{-}$. But, we have $r^{-} \geq 2^{(n+1)}$. Thus, clearly N_2 such that for $n \geq N_2$ $2^{(n+1)} > n^4$. Therefore, $n \geq N_2$ $r < 2^{-N_2-1}$ we have (1) and the claim is proven.

Then $\mu(V(t)) < \frac{1}{t}$.

We will show that $T^{-i}V(t)$ $T^{-j}V(t)$ for $i = j$, $0 \le i \le t$ and therefore, since T is μ preserving, each has the same measure $\mu(V(t)) < \frac{\mu(X)}{t} = \frac{1}{t}$. Let $0 \le j < i \le t$. Suppose x $T^{-i}V(t)$ $T^{-j}V(t)$. Then, we have $T^{i}x$ $V(t)$ V and $T^{j}x$ $V(t)$ V. Therefore, if $y = T^{i}x$, y V but $T^{j-i}y$ $V(t)$ V and $j - i < t$ therefore, $y = T^{i}x\ /\ V(t)$, a contradiction. Thus, we have proven the first claim.

Claim 2: Let $m_\alpha(Y) < c < \infty$. Then > 0 and $p \ge 1$ a measurable set $F = F(p, \ )$ X, with $\mu(F) > 1 - \frac{1}{p}$, such that x F an integer k such that

$$d(f(x), f(T^{k}x)) < \min\left(\left(\frac{4cp^2}{k}\right)^{\frac{1}{\alpha}}, \ \right)$$

By the definition of the Hausdor α measure, we can find a countable cover of $Y = \ _{i\ge1} U_i$, with U_i having $\mathrm{diam}(U_i) = r_i < \min(1, \ )$ and $_{i\ge1} r_i^{\alpha} < c$. Without loss of generality, we may assume U_i are Borel and disjoint up to sets of measure 0. We may assume the sets are Borel since for every set U Y with $m_\alpha(U) < \infty$ U W Y such that W is Borel and $m_\alpha(U) = m_\alpha(W)$. We then make the sets disjoint by taking $\tilde{U}_i = U_i \setminus \ _{1\le j<i} U_i$, which will still be Borel.

Now, denote $V_i = f^{-1}(U_i), v_i = \mu(V_i)$. We examine the set

$$J = \{ \lambda$$

□
$_{i \notin J}\ \mu$

Now, clearly if $T^K x \notin F(p)$ then, $\forall n > K$ $T^n x \notin F(p)$. Therefore, $\forall n > \inf_{x \in F(p)}\{K_x\} > 0$, we have $A \subseteq T^{-n}(X \setminus F(p)) \Rightarrow \mu(A) \leq \frac{1}{p}$ since T is measure preserving. Therefore, since $\mu(F'(p)) = \mu(F(p) \setminus A) \geq \mu(F(p)) - \mu(A) \geq 1 - \frac{2}{p}$ as desired.

Claim 4: Completion of the proof for $m_\alpha(Y) < \infty$.
Let $F' = \bigcup_{p \geq \max(1, \frac{1}{4c})}$. Then by claim 4, $\mu(F') = 1$ and $\forall x \in F$, (4) holds.

Now, if $m_\alpha(Y) = 0$, then we show that 3 holds. Let m

Now,

$$E \setminus E' = \left\{ x \in K \,|\, \liminf_{k\geq 1}\{k^{\frac{1}{\alpha}} d(f(x), f(S^k(x)))\} < \infty \text{ and } \liminf_{n\geq 1}\{n^{\frac{1}{\alpha}} d(f(x), f(T^n(x)))\} = \infty \right\}.$$

But $S^k = T^{nk}$ therefore, if

$$\liminf_{k\geq 1}\{k^{\frac{1}{\alpha}} d(f(x), f(S^k(x)))\} < \infty$$

then

$$\liminf_{k\geq 1}\{n(x)^{\frac{1}{\alpha}} k^{\frac{1}{\alpha}} d(f(x), f(T^{nk}(x)))\} < \infty$$

and therefore,

$$\liminf_{n} \geq 1\{n^{\frac{1}{\alpha}} d(f(x), f(T^n(x)))\} < \infty.$$

Thus, $\mu(E \setminus E') = 0 \Rightarrow \mu(E) = 0$, a contradiction. Thus, the reduction is complete.

5.2 Local Theorems

We will follow [1] to prove these theorems. We will need the following lemmas to prove the theorems on local recurrence behavior.

Lemma 4 *Let μ be a finite Borel measure on the separable metric space X, and $G \subset supp\mu$ a measurable set. Given $r > 0$, $\exists$ a countable set $E \subset G$ such that*

1. $B(x, r) \bigcap B(y, r) = \emptyset$ *for any two distinct pointx $x, y \in E$*
2. $\mu(G \setminus \bigcup_{x\in E} B(x, 2r)) = 0$

Proof:
Order the the collection of subsets of G satisfying the first property by inclusion. Clearly this collection is nonempty since any single point set in G is in it. Then by Zorn's lemma since G is an upper bound for the collection, $\exists$ a maximal set $E \subset G$. Now, since $\mu(B(x, r)) > 0$ for each $x \in E \subset supp\mu$, the set E is at most countable.

Lemma 5 *Let (X, N, μ, T, d) be an m.m.p.s. with T Borel measurable. Then if μ is w.d.r. on a measurable set $Z \subset X$ with $\mu(Z) > 0$, (6) holds for μ-almost every $x \in Z$.*

Lemma 6 *Given $x \quad X$, we have $R^l(x) \leq d \Leftrightarrow$ for every $\quad > 0$,*

$$\liminf_{n\to\infty} n^{\frac{1}{d+}} d(T^n x, x) = 0 \tag{11}$$

Proof:

Assume that $R^l(x) \leq d$. Then given $\quad > 0 \quad (r_n)$ such that $r_n \to 0$, and $\tau_{r_n}(x) < r_n^{-(d+\ )}$ for all n. Let $m_n = \tau_{r_n}(x)$. If m_n is bounded, then x is periodic and clearly (11) holds. Now, if m_n is unbounded, we have $d(T^{m_n} x, x) < r_n$ and

$$\begin{aligned} m_n^{\frac{1}{d+2}} d(T^{m_n} x, x) &< \tau_{r_n}(x)^{\frac{1}{1+2}} r_n \\ &< r_n^{-\frac{d+}{d+2}} r_n = r_n^{\overline{d+}}. \end{aligned}$$

Therefore

$$\liminf_{n\to\infty} n^{\frac{1}{d+2}} d(T^n x, x) \leq \liminf_{n\to\infty} m_n^{\frac{1}{d+2}} d(T^{m_n} x, x) = 0.$$

Thus, (11) holds for every $\quad > 0$.

Now, assume (11) holds for all $\quad > 0$. Let $r_n = 2d(T^n x, x)$. We have that $\tau_{r_n}(X) \leq n$, and thus

$$\liminf_{n\to\infty} \tau_{r_n}(x)^{\frac{1}{d+}} r_n = 0.$$

Thus, a divergings sequnce of positive integers k_n such that $\tau_{r_{k_n}}(x)^{\frac{1}{d+}} r_{k_n} < 1$ for every n. Therefore,

$$R^l(x) \leq \liminf_{n\to\infty} \frac{\log \tau_{r_n}(x)}{-\log r_n} \leq \liminf_{n\to\infty} \frac{\log r_{k_n}^{d+}}{-\log r_{k_n}} = d + \ .$$

Since was arbitrary, we have our result.

Now, to prove theorem 4 we simply apply lemma 5.

To prove theorem 5 we apply theorem 4 and lemma 6.

We will have to prove something more to prove the strongest result, theorem 6.

Proof:

By theorem 4 we have $R^l(x) \leq d^l_\mu(x)$ and $R^u(x) \leq d^u_\mu(x)$ for μ-almost every $x \quad X$. We need to obtain the reverse inequalities.

Since μ is w.r.d. and μ has long return time with respect to T and $d^l_\mu(x) > 0$ for μ-almost every $x \quad X$, if $\quad > 0$ is small enough, we have that $a, \gamma, \rho > 0$ and $G \quad X$ with $\mu(G) > 1 -$ such that if $x \quad G$ and $r \quad (0, \rho)$

$$\mu(A\ (x, 2r)) \leq \mu(B(x, 2r))^{1+\gamma}, \tag{12}$$

$$\mu(B(x, 2r)) \leq \mu(B(x, \frac{r}{2}))r^{-a\frac{\gamma}{2}}, \tag{13}$$

$$\mu(B(x, r)) \leq r^a \tag{14}$$

where $A\ (x, 2r)$ is as in definiton 9. Now, consider

$$A\ (r) := \{y \quad G | \tau_r(y) \leq \mu(B(y, 3r))^{-1}\ \}.$$

Then, if $d(x, y) < r(a)$, we have $\tau_r(y, y) \geq \tau_{2r}(y, x)(b)$. Then, since $B(x, 2r)$ $B(y, 3r)$, if $x \quad G$ then we obtain

$$\begin{aligned}
\mu(B(x, r) \ \square\ A\ (r)) &\leq \mu(\{y \quad B(x, r) | \tau_{2r}(y, x) \leq \mu(B(x, 3r))^{-1+}\ \}) && (a), (b) \\
&\leq \mu(A\ (x, 2r)) && \mu(B(x, 3r)) \geq \mu(B(x, 2r)) \\
&\leq \mu(B(x, 2r))^{1+y} && (12) \\
&\leq \mu(B(x, \tfrac{r}{2}))r^{-a\frac{y}{2}}(2r)^{ay} && (13)(14)
\end{aligned}$$

Then, if $E \quad G$ is a maximal $\frac{r}{2}$-separated set given by lemma (4), we have

$$\begin{aligned}
\mu(A\ (r)) &\leq \ \square\ \square_{x \in E}\ \mu(B(x, r)\ \square\ A\ (r)) \\
&\leq \quad {}_{x \in E}\ \mu(B(x, \tfrac{r}{2}))r^{-a\frac{y}{2}}(2r)^{ay} \quad . \\
&\leq \quad 2^{ay} r^{a\frac{y}{2}}
\end{aligned}$$

Here, the last step follows from (14). Then, the Borel-Cantelli lemma gives us that for $\mu-$almost every $x \quad G$ we have

$$\tau_{e^{-n}}(x) > \mu(B(x, 3e^{-n}))^{-1+}$$

for all $n \quad B$0 T(5 0 (Td [(026)]TJ/F15 9.962

6.2 Baker's Map

Let $T : [0,1]^2 \to [0,1]^2$ be the standard bakers map. We then have that T is Lebesgue measure preserving. And thus, by theorem 3 we have that for Lebesgue almost every $x \in [0,1]^2$, we have

$$\tau_r(x, x) \leq C_x r^{-2}$$

for r small enough. Further, by theorem 4, since $d_\mu^u(x) = d_\mu^l(x) = 2$ for μ the Lebesgue measure, $\tau_r(x, x) \geq r^{-2}$. So, we have bounded the recurrence time for

References